Homework Due Thursday May 12th Instead of a Quiz

Part 1 - Redo the Midterm

Rewrite the solutions to one of the exams. You can do either Version A or Version B.
• You must copy down each problem again prior to writing its solution.
• It's fine to use the posted solutions for the respective exams, or you can redo the exam again on your own. The purpose of this is to go back and really understand the questions and their solutions. * The solutions are on the main course webpage *
• Please do not treat it as just a menial copying task.

Part 2 - Extra Problem

Complete the following proof. It is a generalization of number 4 of the Practice Midterm. Please copy this problem, too, but you don't have to copy it word for word since it's long and I added comments.

The Problem: (Here, k is a constant)
. Let $f(x,y,z) \equiv k$ be a level surface (For example see Pg 830 in 12.6. Some specific examples: a sphere is $f(x,y,z) = x^2+y^2+z^2 = k^2$, or a paraboloid is $f(x,y,z) = z-x^2-y^2 = k$.)
. Let the vector field $\mathbf{F}(x,y,z) = \nabla f(x,y,z) =< f_x, f_y, f_z >$.
(We could take $\mathbf{F}$ to be a scalar multiple of ∇f, i.e. $\mathbf{F} = c\nabla f$ but let's not for simplicity).
. Let C be any (piecewise smooth) curve given by $\mathbf{r}(t) =< x(t), y(t), z(t) >, a \leq t \leq b$ where C lies on the surface $f(x,y,z) \equiv k$. In other words, $f(x(t),y(t),z(t)) = k$ at any time t.
Show/Prove that: $\int_C \mathbf{F} \cdot \mathbf{dr} = 0$.

Outline of the proof:
a) Start with writing $\int_C \mathbf{F} \cdot \mathbf{dr}$ as an integral in time.
b) Since the curve lies on the surface, $f(x(t),y(t),z(t)) = k$ for all times t, we have that since the RHS k is a constant, $\frac{d}{dt} f(x(t),y(t),z(t)) = 0$. Specifically, by the chain rule,

$$0 = \frac{d}{dt} f(x(t),y(t),z(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

c) From (a) and (b), conclude that $\int_C \mathbf{F} \cdot \mathbf{dr} = 0$.

How this relates to #4 on the Practice MT:
Since on that problem, $|\mathbf{r}(t)| = \textit{constant}$, this means the curve lies on some sphere centered at the origin! Notice that a sphere is then $f(x,y,z) = x^2+y^2+z^2 = R^2$ and that $\nabla f =< 2x, 2y, 2z >$ so $\mathbf{F} = \frac{1}{2}\nabla f$ in his problem. # 4 is a specific example of the above generalization.

The idea is the following. First, $\mathbf{F} = \nabla f$ is perpendicular to the surface since it's the gradient. On the other hand since the curve $\mathbf{r}(t)$ lies on the surface, the tangents to the curve, given by $\mathbf{r}'(t)$, are all tangent to the surface, too. But, this means that "$\mathbf{dr}$" $= \mathbf{r}'(t) \perp \nabla f = \mathbf{F}$ so the dot product of $\mathbf{F} \cdot \mathbf{dr}$ is always zero.

Proof. We start with the definition. Let us also write $f(x(t), y(t), z(t))$ as $f(\mathbf{r}(t))$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_a^b f_x(\mathbf{r}(t)) \cdot x'(t)dt + f_y(\mathbf{r}(t)) \cdot y'(t)dt + f_z(\mathbf{r}(t)) \cdot z'(t)dt$$

$$= \int_a^b [f_x(\mathbf{r}(t))x'(t) + f_y(\mathbf{r}(t))y'(t) + f_z(\mathbf{r}(t))z'(t)]dt.$$

Since the curve lies on the surface $f(x, y, z) = k$, we have for any t, $f(x(t), y(t), z(t)) = k$ so if we take the time derivative, for ANY time t

$$0 = \frac{d}{dt}(constant\ k) = \frac{d}{dt} f(x(t), y(t), z(t)) \overset{Chain\ Rule}{=} \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= f_x(\mathbf{r}(t))x'(t) + f_y(\mathbf{r}(t))y'(t) + f_z(\mathbf{r}(t))z'(t).$$

Since $f_x(\mathbf{r}(t))x'(t) + f_y(\mathbf{r}(t))y'(t) + f_z(\mathbf{r}(t))z'(t) = 0$ at any time t, this means the integrand of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is always zero, i.e.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b f_x(\mathbf{r}(t)) \cdot x'(t)dt + f_y(\mathbf{r}(t)) \cdot y'(t)dt + f_z(\mathbf{r}(t)) \cdot z'(t)dt = \int_a^b 0dt = 0.$$

(And we have proved the statement).

$\square$